Isothermal Flows in Water

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Nomenclature

r = Eulerian coordinate of the particle
r_0 = Lagrangian coordinate of the particle
r_2 = shock coordinate
t = time
v = velocity of the particle
p = pressure of the particle
T = temperature of the particle
ρ = density of the particle
ρ_* = characteristic density
γ = constant

Introduction

BLAST wave theory, proposed by Taylor[1] and Sedov,[2] deals with the assumption that the flow behind the shock is adiabatic. The numerical solutions obtained by them indicate that the temperature behind the shock increases and leads toward infinity at the center of symmetry of the blast. Such large temperature gradients give rise to conduction currents that should be taken into consideration. Because of heat conduction, intensive heat exchanges take place between the particles and hence the adiabatic condition does not remain valid. However, an alternative assumption that the shock engulfed region is isothermal, i.e., temperature gradient is zero, appears to be more reasonble.

Korobeinikov[3,4] investigated for the first time the isothermal flows behind strong shock in an ideal gas for both uniform and nonuniform medium. Following the numerical method due to Korobeinikov,[3] Kot[5] investigated isothermal flows in water to study the initial phase of the intense point explosion. Since the point explosion problem in water is of considerable importance to ordnance scientists (see Cole[6]), an attempt is made to obtain analytic solutions to this problem for the initial phase of the explosion when the flow is self-similar. The shock wave initiated by an intense point explosion is assumed to be strong, propagating in still water. Flow behind the shock is considered to be isothermal. An integral method developed by Laumbach and Probstein[7] is adopted to investigate approximate analytic solutions. The solutions are obtained in closed analytic form. Comparison of these solutions with the exact numerical solutions of Kot[5] shows excellent agreement.

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Basic Equations and Boundary Conditions

The basic equations of continuity and momentum governing the flowfield behind the spherical shock are taken in Lagrangian form as

$$\rho r^2 \mathrm{d}r = \rho_1 r_0^2 \mathrm{d}r_0 \tag{1}$$

$$\partial^2 r/\partial t^2 = -(1/\rho)\ (\partial p/\partial r) \tag{2}$$

where r_0 is the initial coordinate of the particle and ρ_1 is the ambient density. The isothermal condition is characterized by the equation

$$\partial T/\partial r = 0 \tag{3}$$

The equation of state for water is taken in the form

$$p = \psi(T)[(\rho/\rho_*)^\gamma - 1] \tag{4}$$

where ρ_* is the characteristic density.

Introducing the parameter $\beta \equiv \rho_1/\rho_2$, the strong shock conditions can be written as

$$v_2 = (1-\beta)\dot{r}_2 \tag{5}$$

$$p_2 = (1-\beta)\rho_1 \dot{r}_2^2 \tag{6}$$

where r_2 is the shock radius and $\dot{r}_2$ is the shock velocity. The overdot denotes differentiation with respect to time t. The suffix 2 denotes quantities just behind the shock.

Method and Solutions

Here we follow the integral method of Laumbach and Probstein[7] and omit the explanations for the assumptions involved in it. The momentum equation (2) can be written in integral form as

$$p(r_0,t) = p_2(r_2) + \int_{r_0}^{r_2} \frac{1}{r^2}\frac{\partial^2 r}{\partial t^2}\rho_1 r_0^2 \mathrm{d}r_0 \tag{7}$$

The integral in Eq. (7) can be evaluated, approximately, by replacing the integrand $[(1/r^2)\ (\partial^2 r/\partial t^2)]$ by its value at the shock (see the Appendix). We obtain

$$p(r_0,t) = p_2(r_2) + \frac{\partial^2 r}{\partial t^2}\Big|_2 \frac{\rho_1 r_2}{3}(1-\xi^3) \tag{8}$$

where $\xi = r_0/r_2$ is the reduced Lagrangian coordinate. Now the pressure distribution can be obtained from Eq. (8) provided the shock propagation law is known. We then take this propagation law as $r_2 = C\ t^{2/5}$, where C is constant, established by the dimensional considerations (see Sedov[2] and Korobeinikov).[4] Making use of this relation, Eqs. (A11) and (6), we obtain from Eq. (8)

$$p(r_0,t)/p_2(r_2) = 1 + H(1-\xi^3) \tag{9}$$

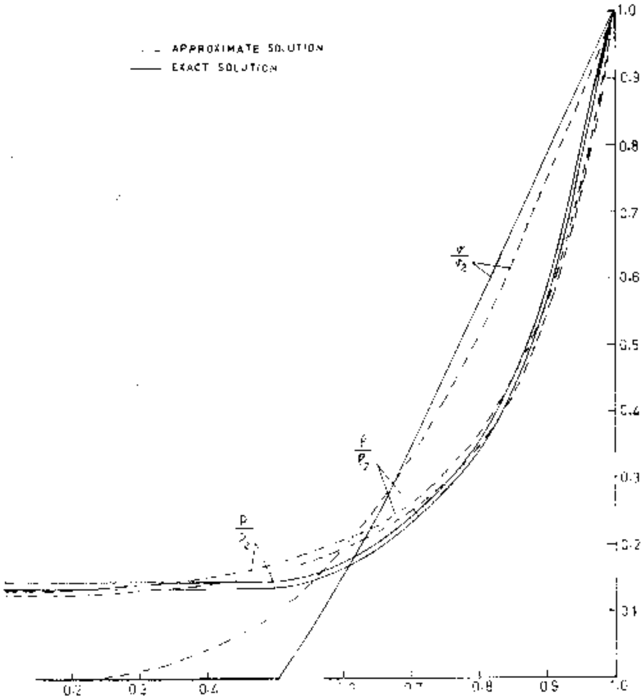

Fig. 1 Comparison of approximate analytic solutions and the exact numerical solutions of Kot.[5]

where

$$H = M(1-\beta)(4\beta-3)/6[M-\beta(1+M)] \tag{10}$$

$$M = \gamma(G/\beta)^{\gamma}/[(G/\beta)^{\gamma}-1], \quad G = \rho_1/\rho_* \tag{11}$$

The equation of continuity (1) can be written in integral form as

$$\int_0^r r^2 \mathrm{d}r = \int_0^{r_0} \frac{\rho_1}{\rho} r_0^2 \, \mathrm{d}r_0 \tag{12}$$

Introducing the reduced Eulerian and reduced Lagrangian coordinates, Eq. (12) can be written as

$$\lambda^3 = 3\beta \int_0^{\xi} \frac{\rho_2}{\rho} \xi^2 \mathrm{d}\xi \tag{13}$$

where $\lambda = r/r_2$ is the reduced Eulerian coordinate. It takes the value 1 at the shock and zero at the center of symmetry. Equations (3) and (4) yield

$$\frac{p(r_0,t)}{p_2(r_2)} = \frac{[\rho(r_0,t)/\rho_*]^{\gamma}-1}{[\rho_2/\rho_*]^{\gamma}-1} \tag{14}$$

Substitution of p/p_2 from Eq. (9) into Eq. (14) yields

$$\frac{\rho(r_0,t)}{\rho_2} = \frac{\beta}{G}[R\{1+H(1-\xi^3)\}+1]^{1/\gamma} \tag{15}$$

where $R = (G/\beta)^{\gamma}-1$. Making use of Eq. (15) into Eq. (13), we obtain

$$\lambda^3 = \frac{\gamma G}{(\gamma-1)HR}$$

$$\{[(1+H)R+1]^{(\gamma-1)} - [\{1+H(1-\xi^3)\}R+1]^{\frac{(\gamma-1)}{\gamma}} \tag{16}$$

Since at the shock $\xi = 1$ and $\lambda = 1$, Eq. (16) yields

$$(\gamma-1)HR - \lambda G[\{(1+H)R+1\}^{(\gamma-1)/\gamma} - (G/\beta)^{(\gamma-1)}] = 0 \tag{17}$$

The value of β for known values of the parameters G and γ can be obtained from Eq. (17). Differentiating Eq. (16) with respect to time t and using Eq. (5), we have

$$\frac{v(r_0,t)}{v_2(r_2)} = \frac{1}{(1-\beta)\lambda^2}\{\lambda^3 - G[\{1+H(1-\xi^3)\}R+1]^{1/\gamma}\xi^3\} \tag{18}$$

It is interesting to observe that the solutions p/p_2, ρ/ρ_2 and v/v_2 so obtained are in closed analytic form. A relationship between the reduced Eulerian and reduced Lagrangian coordinated can be obtained from Eq. (16). Comparison of approximate analytic solutions with the exact numerical solutions of Kot[5] shows good agreement (Fig. 1). The parameters involved in the solutions are taken as $\gamma = 0.95$, $G = 82.074$ and $\rho_1 = 1$ (see Ref. 5). From Fig. 1, one can observe the formation of a core near the center of symmetry in which there is no flow. The difference in the extent of the cores predicted by approximate and exact numerical solutions may be attributed to the assumption made in the present analysis in obtaining approximate solutions. Flow variables are assumed, in our analysis, to be analytic functions of Eulerian coordinates and hence do not admit discontinuities such as those occurring near $\lambda = 0.5$ in the exact numerical solutions.

Appendix

We first write Taylor's expansion for $r(r_0,t)$ as

$$r(r_0,t) = r_2 + (\partial r/\partial r_0)|_2(r_0-r_2)$$

$$+ \frac{1}{2}(\partial^2 r/\partial r_0^2)|_2(r_0-r_2)^2 + \ldots \tag{A1}$$

After introducing the parameter $\beta \equiv \rho_1/\rho_2$, the equation of continuity (1) yields

$$(\partial r/\partial r_0)|_2 = (\rho_1 r_0^2/\rho r^2)|_2 = \beta \tag{A2}$$

Equations (A1) and (A2) yield

$$(\partial^2 r/\partial t^2)|_2 = (1-\beta)\ddot{r}_2 + (\partial^2 r/\partial r_0^2)|_2 \dot{r}_2^2 \tag{A3}$$

Using the shock condition (6), the momentum equation (2) can be written as

$$\frac{\partial^2 r}{\partial t^2}\Big|_2 = -(1-\beta)\dot{r}_2^2\left(\frac{1}{p}\frac{\partial p}{\partial r_0}\right)\Big|_2 \tag{A4}$$

Equating Eqs. (A3) and (A4) we obtain

$$-\frac{\partial^2 r}{\partial r_0^2}\Big|_2 \dot{r}_2^2 = (1-\beta)\ddot{r}_2 + (1-\beta)\dot{r}_2^2\left(\frac{1}{p}\frac{\partial p}{\partial r_0}\right)\Big|_2 \tag{A5}$$

Logarithmic differentiation of the continuity equation (1) yields

$$\left(\frac{1}{\rho}\frac{\partial\rho}{\partial r_0}\right)\Big|_2 = \frac{2(1-\beta)}{r_2} - \frac{1}{\beta}\frac{\partial^2 r}{\partial r_0^2}\Big|_2 \tag{A6}$$

Logarithmic differentiation of the equation-of-state (4) yields

$$\left(\frac{1}{p}\frac{\partial p}{\partial r_0}\right)\Big|_2 = M\left(\frac{1}{\rho}\frac{\partial\rho}{\partial r_0}\right)\Big|_2 \tag{A7}$$

where

$$M=\gamma(G/\beta)^{\gamma}/[(G/\beta)^{\gamma}-1] \tag{A8}$$

$$G=\rho_1/\rho. \tag{A9}$$

Substituting Eq. (A6) into (A7) and making use of Eq. (A5) we obtain

$$\frac{\partial^2 r}{\partial r_0^2}\Big|_2 \dot{r}_2^2=\frac{\beta(1-\beta)}{M-\beta(1+M)}\left[\ddot{r}_2+2M(1-\beta)\frac{\dot{r}_2^2}{\dot{r}_2}\right] \tag{A10}$$

Substitution of Eq. (A10) into Eq. (A3) yields

$$\frac{\partial^2 r}{\partial t^2}\Big|_2=\frac{M(1-\beta)^2}{M-\beta(1+M)}\left[\ddot{r}_2+2\beta\frac{\dot{r}_2^2}{r_2}\right] \tag{A11}$$

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